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The city of Strasbourg is the home of many architectural masterpieces, and in the first place the magnificent cathedral. Its spire was completed in 1439, but some of its parts were added during the later Renaissance. The case of today's astronomical clock, for instance, was built between 1571 and 1574, and its stone structure, as well as the winding staircase that allows access to its various mechanisms, were built by Hans Thomann Uhlberger, the then architect of the cathedral.

Uhlberger was the master architect of the *Œuvre Notre-Dame*, the institution in charge of the construction and maintenance of the cathedral. And the building of the *Œuvre Notre-Dame* itself contains another most remarkable staircase by Uhlberger, dated around 1580, that one author called the "queen of spiral staircases" [18]. This is a winding staircase with a hollow central newel (figures 1 and 2). A visit to the *Musée de l'Œuvre Notre-Dame* will lead through it, and it should not be missed.

Figure 1: The *Musée de l'Œuvre Notre-Dame* in Strasbourg. The staircase is located within the hexagonal tower between the two buildings. (Wikipedia)

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Figure 2: The upper part of the staircase in the *Musée de l'Œuvre Notre-Dame* in Strasbourg. (Wikipedia)

1 The circular structures of the newel

The staircase of the *Œuvre Notre-Dame* is a sinistral (left-handed) staircase with a hollow newel. More precisely, the newel is made of three columns which are joined at the top by two circular constructions (figure 2).

The lowest of the circular constructions is a relatively simple sector, with grooves reminiscent of other parts of the staircase. This sector joins the three columns, one of which has a corinthian capital.

The three columns extend above this sector and reach a fully circular structure, before continuing to the sexpartite-like rib vault. This last circular structure, which is in fact the end of the ramp, is one of the most interesting features of the staircase.

A close examination reveals that the curves which make up this circular structure are nearly circular rings. (We assume "ring" to mean an annular shape, but not necessarily a circular one.) The fact that the structure contains rings was of course known by Uhlberger and those who studied the staircase afterwards, but this feature got renewed interest a few years ago when the French mathematician Marcel Berger interpreted these rings as so-called "Villarceau circles" [1, 2, 3, 4, 5, 6, 7]. We will go into more detail about these rings in the next section, but for now, what is important for us is that these rings are interlaced and non-intersecting. These rings were again mentioned in the recently published guide of the museum [11, p. 179], albeit without crediting Berger.

How many rings does this structure contain? Berger identified three rings (figure 3, left) [4, 7].

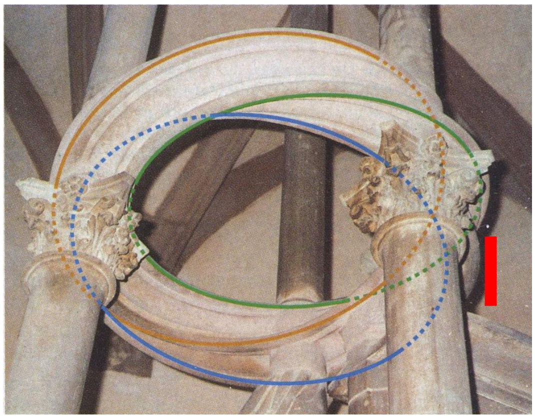


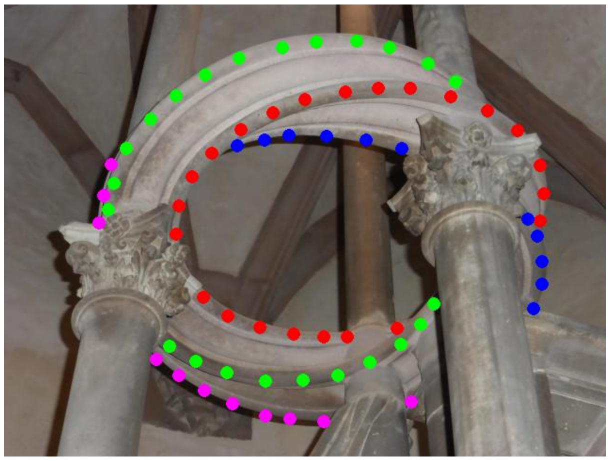
Figure 3: Two pictures used in Berger's article [4].

It is true that the photograph used by Berger seems to show three circles, but the picture contains at least one inconsistency, namely that there seems to be something else towards the right, which we marked in red in figure 3. Moreover, Berger's 2002 article has four *different* pictures of the circular structure, and the one in figure 3 (right) is even more explicit.

There are in fact not three, but *four* rings in that circular structure! The fact that Berger missed the fourth ring may be due to the assumption that the number of rings equals the number of columns in the newel, or to an indirect examination. And the fact that there are three columns is itself determined by the rib vault, and hence by the number of sides of the tower containing the staircase.

Figure 4 shows another view of the circular structure, and an identification of these four rings. It appears that Berger merged two circles, and missed the circle protruding towards the right. The fact that the blue circle (in our reconstruction and in Berger's picture) cannot be seen from below is obvious when the circular structure is examined from above (figure 5).

(a)



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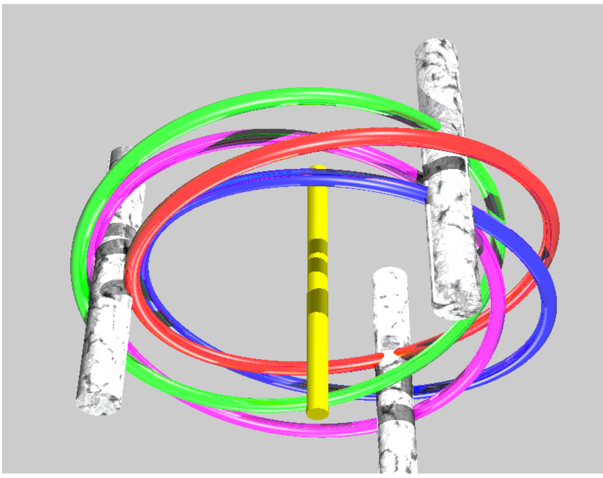

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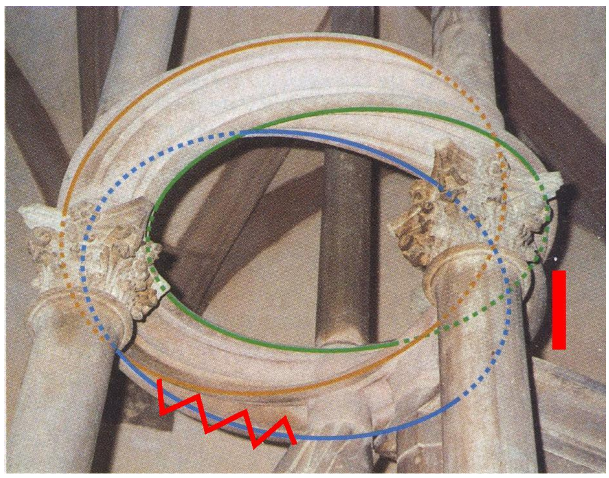
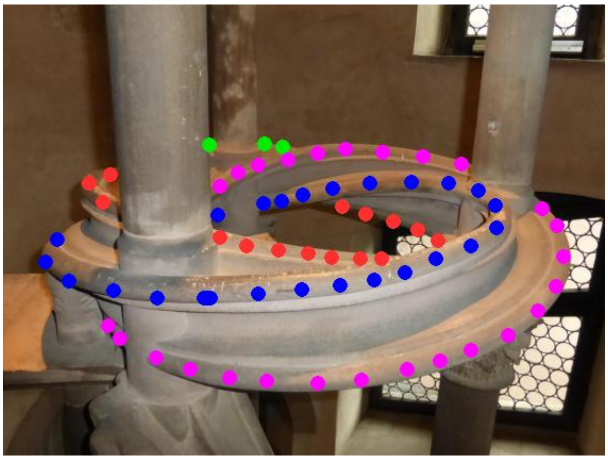
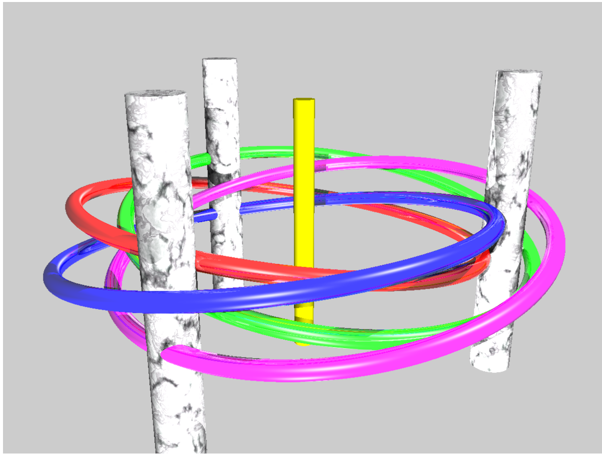


(d)

Figure 4: (a) The interlaced rings seen from below. (Photograph by the author.) (b) The identification of the four rings. (c) A reconstruction of the rings, using tilted circles, offset, and rotated 90, 180 and 270 degrees around the yellow axis. (d) The picture used in Berger's article [4], with the blue circle incorrectly drawn. The wrong part is marked in red, and corresponds to the fourth circle.



(a)



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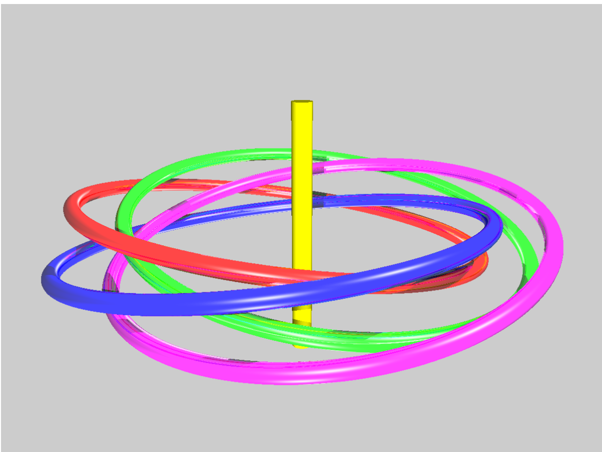

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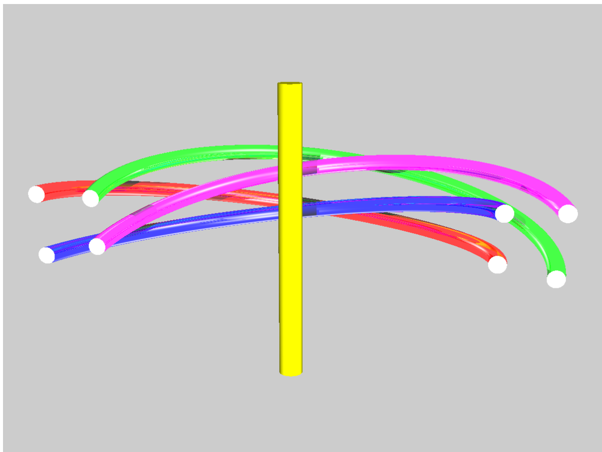

(d)

Figure 5: (a) The interlaced rings seen from above and identified. Three rings are clearly visible, and small parts of the green ring can be guessed. (Photograph by the author.) (b–d) A reconstruction of the rings, with and without the three columns of the newel, and cut by a vertical plane going through the virtual central axis shown in yellow.

2 Villarceau circles

In his lectures on geometry published in 1972–1974 [1], and perhaps before, Berger identified the rings of the top circular structure as "Villarceau circles." Berger's statement was repeated in subsequent editions of his lectures, in particular in 1977 [2] and 1990 [3]. In all these works, Berger included a picture drawn from Hans Haug's book [12]. Haug was the curator of the Strasbourg museums and the founder of the museum of the *Œuvre Notre-Dame*. Berger's (brief) description of the staircase was reprinted (with the picture) in the journal *L'Ouvert* in 1998 [15], and it eventually made its way to a larger audience in 2002 [4] and a few years later in Berger's *Géométrie vivante* [5, 6]. Berger's 2002 text was further expanded in 2010 for an online text on the *bibnum* site [7]. Only in 2002 and 2010 did Berger consider that there are three circles. The number of circles is apparently not mentioned in Berger's other works.

Berger's claim is that the top of the staircase is a torus sculpted in such a way that its edges are Villarceau circles. This analysis, in turn, is used to support the claim that these circles were known long before Villarceau.

So, what are these "Villarceau circles"? Villarceau circles are certain circles which can be traced on a torus. A torus is the surface generated by a circle rotating around an axis lying in the circle's plane and not intersecting it (see figure 6). Through each point of the torus, it is possible to find four circles lying on the torus. One circle has its axis parallel to the generation axis, another one is the generating circle, and the last two are known as Villarceau circles, as they were first described by Antoine Yvon Villarceau (1813–1883) in 1848 [20].

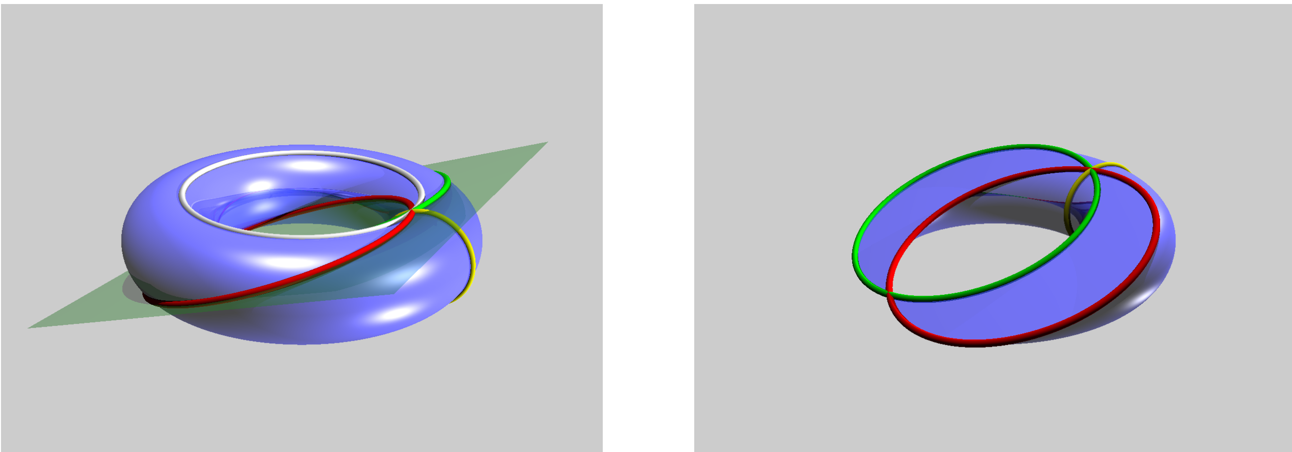

Figure 6: A torus with two of the circles discovered by Villarceau (in red and green), in addition to the white and yellow circles which were well known before Villarceau. The view on the right shows the lower half of the torus cut by the plane containing two of the Villarceau circles.

Conversely, if a circle is rotated, it may generate a torus. However for that to be the case, the offset of the circle needs to be related to its tilt and to its radius, so that most tilted and offset circles will not generate a torus, although they will generate a similar surface. The circle of radius r first has to be tilted by an angle α around the axis X, then offset along that same axis by $s = r \sin \alpha$, before eventually being rotated around the axis Z in order to generate the torus.

So, one should be cautious, because if the rings of the staircase are planar and circular, it does still not entail that these rings are Villarceau circles of some imaginary torus. Figure 7 shows the surfaces generated by correctly offset circles (a), and those generated by other circles (b–d).

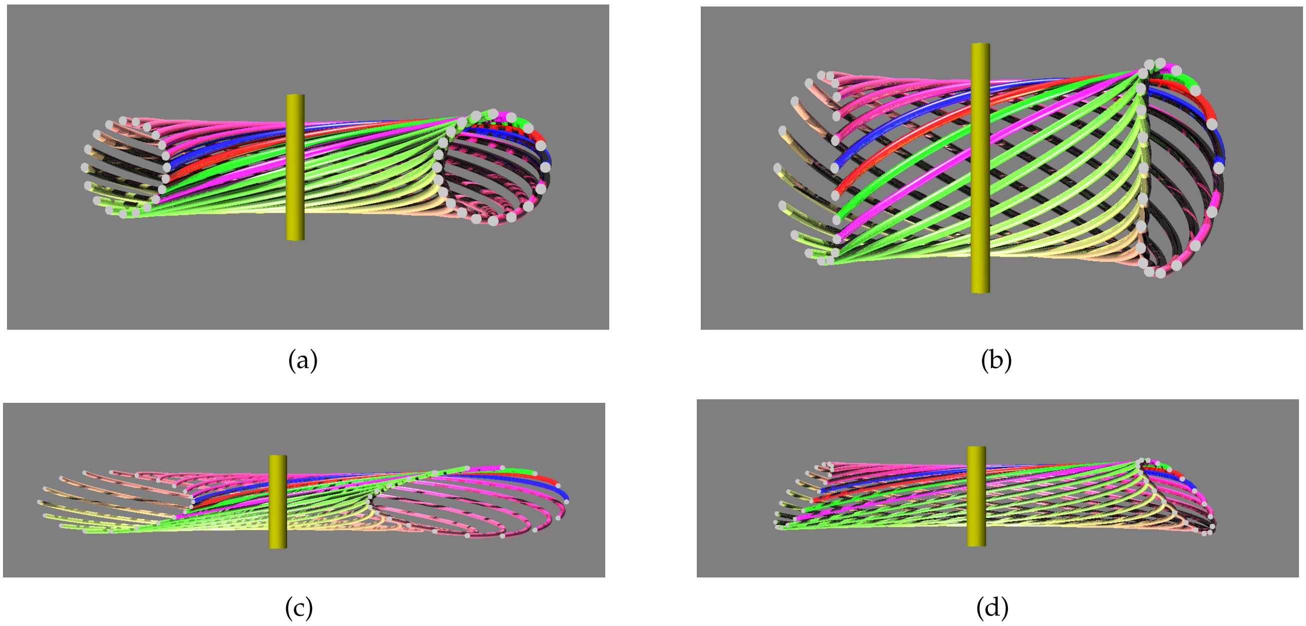

Figure 7: The section of a set of 20 Villarceau circles is shown in (a) and it is circular, although one can observe that the intersections of the circles are not evenly spaced. But when the tilt of the circles is not the right function of its offset, the section of the revolution surface is not a circle, hence the circles are not Villarceau circles. In (b), the tilt is too large. In (c), the tilt is too small. And in (d), the absolute value of the offset is correct, but not in the right direction.

Now that we know what Villarceau circles are, can we conclude about the rings in the staircase?

The first thing that we should notice is that the upper circular structure does not contain any obvious torus. There are rings, and these rings are joined, but the structure does not have the appearance of a torus with superimposed rings. Of course, we can consider that the initial piece of stone was some kind of torus, and that the rings were carved out, but that does not mean that the stone cutter or architect had the understanding that the rings could embrace a torus.

The second observation is that such carved out rings can be designed as rotated rings, that is, one ring can be obtained by the rotation of the previous one. In that case, the architect may have found out that by shifting the tilted circles and rotating them, these circles would not intersect. But that again does not entail that there was a knowledge of an underlying torus.

And finally, we have seen that rotated circles do only produce a torus when the offset is appropriately related to the tilt of the circles.

In summary, nothing in this circular structure supports that these four rings are Villarceau circles of some imaginary torus. They may be such circles, but that can only be ascertained by proper measurements. It seems very unlikely that the rings do exactly lie on a torus.

But if that is nevertheless the case, should we conclude that Uhlberger knew about Villarceau circles? If the torus was visible, I would probably answer yes to that question. But since this is not the case, I consider that it is a *historical mistake* to interpret these four rings as Villarceau circles, even if they are (close to) Villarceau circles of an imaginary torus. I can therefore not subscribe to the statement that "these rings are the practical expression of a theory stated five centuries later by the mathematician who gave them his name."[11, p. 179]. Apart from the fact that there are only *three* centuries between Uhlberger and Villarceau, this statement is a historical distortion, a projection of the future into the past.

We have also assumed that the four rings are circles. But are they? Berger wrote that they are planar, *hence* that they are Villarceau circles. We have seen above that this is a hastily drawn conclusion since the planarity is a necessary, but not sufficient, condition for the curves to be Villarceau circles. In fact, when we examined the rings, their planarity was far from obvious! In figure 5(a), for instance, the blue ring seems to have a bent and the projected curve does not seem convex. The same may perhaps be observed with the other rings. In other words, it is very likely that the rings are not even circular, which should definitely exclude the interpretation as Villarceau circles, and certainly the suggestion that Uhlberger may have known about such a property of the torus.

3 Twisted beams

We have seen that if the rings at the top of the staircase are circles, they could be obtained by rotating one of them. If the offset is related to the tilt and radius of the circles, then the circles generate a torus. Uhlberger certainly did not have a torus in mind, but one might argue that he wanted to experiment with rotated circles and came to a construction which was close to the one which would be obtained by starting from a torus and adding a number of Villarceau circles. This assumption, however, does not explain why the rings are not all planar.

There is however another explanation to the construction of the capital of staircase. And that is that the capital is actually an *extension of the ramp*. The staircase is left-handed (sinistral staircase) and the ramp therefore also twists towards the left. At some point the ramp comes very close to the capital and the capital seems to extend the ramp, but with a zero lead. What is more interesting is that the ramp can be seen as a twisted beam with four (helicoidal) edges. And the capital basically shows four edges, twisted towards the left in the continuation of the ramp. This both explains the orientation of the "circles", and their number.

Indeed, a square beam twisted 360 degrees after one turn gives four interlaced curves (figure 8). A pentagonal beam would give five interlaced curves, and so on.

However a beam twisted regularly only provides an approximation of Villarceau circles, because the density of Villarceau circles varies within a section of the torus (see figure 7(a)). In particular, the curves produced by the edges are not planar (figure 9). They are nevertheless a good approximation to circles, as shown in figure 10.

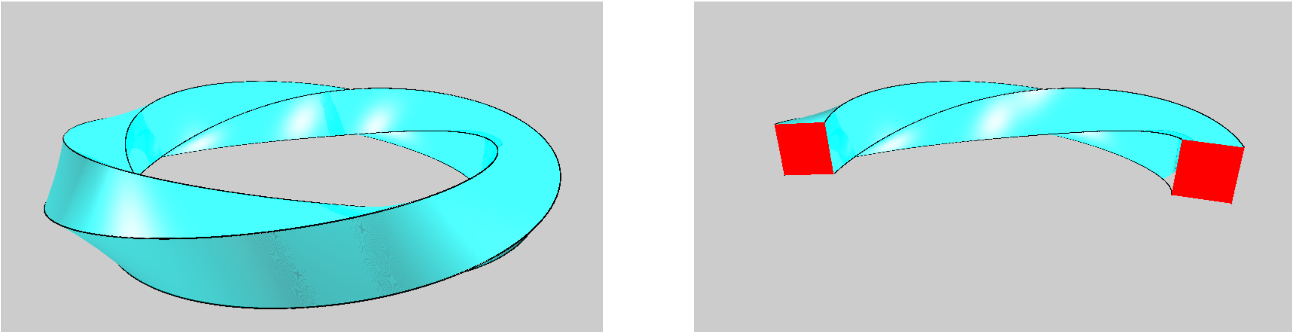

Figure 8: A square beam twisted regularly provides a good *approximation* of Villarceau circles. It is only an approximation, because the density of Villarceau circles in a section of the torus is not constant.

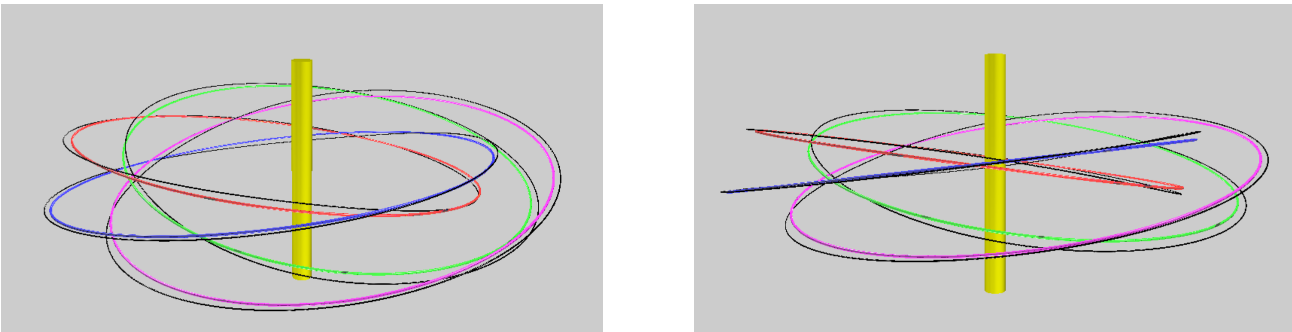

Figure 9: The edges of the twisted beam (in black) superimposed with the Villarceau circles (in color). The two sets of curves are close, but not identical. Moreover, the edges of the twisted beam are not a planar curve, as displayed with the blue circle which lies in the viewing direction.

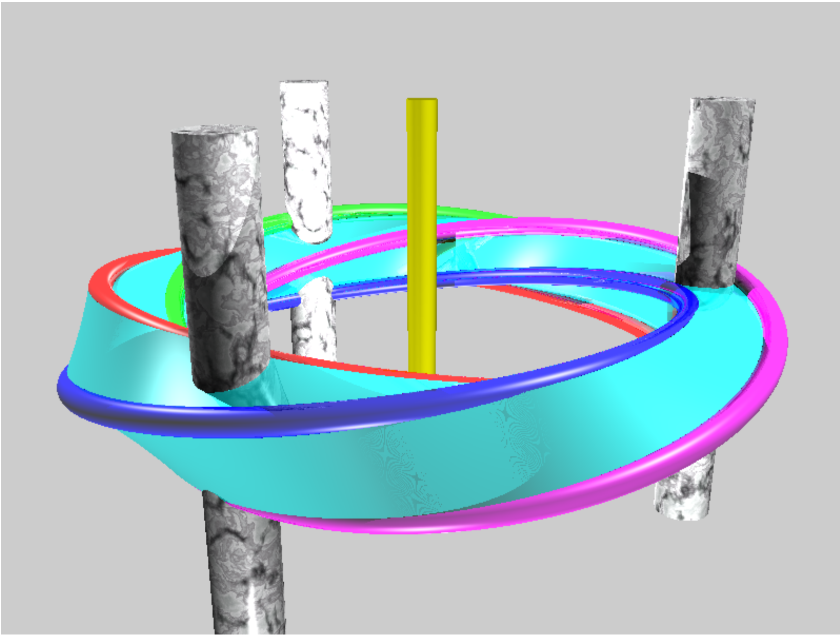

Figure 10: The twisted beam superimposed with the Villarceau circles.

4 Conclusion

The staircase of the *Œuvre Notre-Dame* turns out to be most interesting, and we have discussed the claim that it embeds Villarceau circles. Instead, we consider that the architect Uhlberger had no knowledge about these circles and that these curves (which are only approximate circles) appeared accidentally as an extension of the construction of the ramp of the staircase.

It would now be interesting to have a closer examination of the rings, and in particular to measure the dimensions of the circles. More precisely, the planarity of the four curves should be checked, as well as their size, their tilt, their offset from center, and their spread around the circumference, to ascertain how close these curves are to Villarceau circles of an imaginary torus.

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(References to the works of Czarnowsky, Rumpler and Stoehr were gratefully provided by Mrs. Cécile Dupeux, the curator of the museum.)